Probability and Random Process
Lecture 6

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Review

CDF

$F_{\mathbf{X}}(x) = P\{\mathbf{X} \leq x\}$

PDF

$f_{\mathbf{X}}(x) = \frac{dF_{\mathbf{X}}(x)}{dx}$

Expected value Discrete random variable

$$E(X) = \sum_{\forall x} x f_X(x)$$

Expected value Continuous random variable

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

Moments

Moment about origin

The k^{th} moment about origin of a random variable is defined as $E(X^k)$ where $k \in \{1, 2, \cdots\}$

$$m_k = E(X^k) = \int_{-\infty}^{\infty} x^n f_X(x) dx$$

Central moment

The k^{th} central moment of a random variable is defined as $E((X - E(X))^k)$ where $k \in \{1, 2, \cdots\}$

$$\mu_k = E((X - E(X))^k) = \int_{-\infty}^{\infty} (x - E(X))^n f_X(x) dx$$

What are the values of μ_0 and μ_1 ?

Moment generating functions

The moment generating function of a random variable X is defined as $M_X(\nu) = E(e^{\nu X})$ where ν is a real number such that $-\infty < \nu < \infty$

Moment generating functions

What is e^x

$e^x = \sum_{n=0}^{\infty} x^n/n!$
$e^x = 1/1 + x/1 + x^2/2 + x^3/6 + \ldots$

$e^{\nu X} = 1/1 + \nu X/1 + (\nu X)^2/2 + (\nu X)^3/6 + \ldots$
MGF$= E(1/1 + \nu X/1 + (\nu X)^2/2 + (\nu X)^3/6 + \ldots$

The moments can be found out by successively differentiating the MGF with respect to ν and setting ν to 0.

Moment generating functions

If X is the out come of a fair die, find the MGF of X and using MGF find out expectation of X and variance of X

$P(X = x) = 1/6, x = \{1, \cdots, 6\}$
MGF$= \sum_{X=1}^{6} e^{\nu X} p(X = x)$
$=1/6e^{\nu} + 1/6e^{2\nu} + 1/6e^{3\nu} + 1/6e^{4\nu} + 1/6e^{5\nu} + 1/6e^{6\nu}$

Characteristic functions

The characteristic function of a random variable X is defined as
$\Phi_X(\omega) = E(e^{j\omega X})$ where ω is a real number such that $-\infty < \omega < \infty$
$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x) e^{j\omega X} dx$

The density function can be found out by
$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\omega) d\omega$

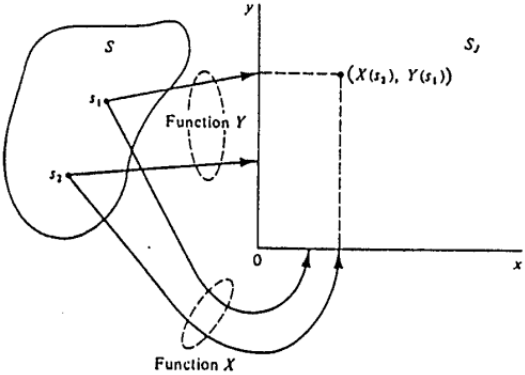
Characteristic functions

The moments can be found out from the characteristic function by differentiating it and setting $\omega = 0$

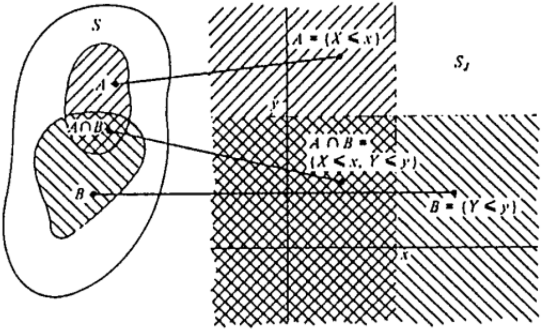
$M_n = (-j)^n \frac{d^n \Phi_X(\omega)}{d\omega^n}|_{\omega=0}$

Multiple random variables

Two random variables



Two random variables



Joint Distribution

We define the event $\{X \leq x, Y \leq y\}$

The joint probability distribution function is
$F_{X,Y}(x, y) = P\{X \leq x, Y \leq y\}$

Joint Distribution Example

Consider a joint sample space with 3 elements (1,1) , (2,1) and (3,3). Assume the probabilities are P(1,1)=0.2, P(2,1)=0.3 and P(3,3)=0.5. Find the joint distribution.

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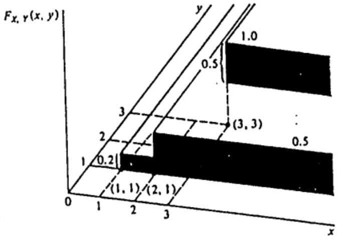
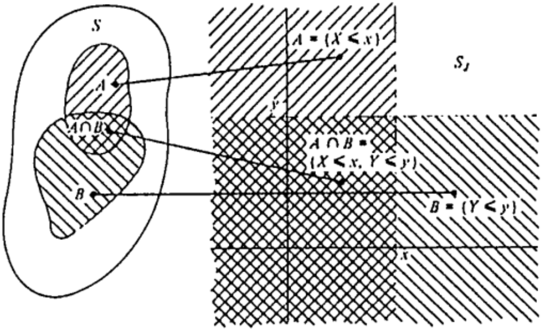


Joint Distribution Properties

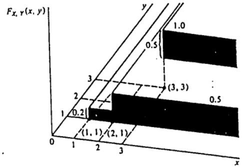
1. $F_{X,Y}(-\infty, -\infty) = 0$
2. $F_{X,Y}(\infty, \infty) = 1$
3. $0 \leq F_{X,Y}(x, y) \leq 1$
4. $F_{X,Y}(x, y)$ is a non decreasing function of both x and y.
5. $F_{X,Y}(x_2, y_2) + F_{X,Y}(x_1, y_1) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1) = P\{x_1 < X \leq x_2, y_1 < Y \leq y_2\} \geq 0$
6. $F_{X,Y}(x, \infty) = F_X(x) \quad F_{X,Y}(\infty, y) = F_Y(y)$

Marginal distribution

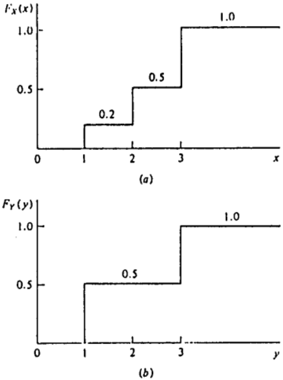
$$F_{X,Y}(x,\infty) = F_X(x) \quad F_{X,Y}(\infty,y) = F_Y(y)$$



Marginal distribution Example



(a)



(b)

N random variables

The joint density of N random variables

$$F_{X_1,X_2,\cdots,X_N}(x_1,x_2\cdots x_N) = P\{X_1 \leq x_1, X_2 \leq x_2 \cdots X_N \leq x_n\}$$

Joint Density function

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

Joint Density function

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