Probability and Random Process
Lecture 5

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August 11, 2015

Overview

1. Review
2. Expected value and Variance

Review

CDF

$F_{\mathbf{X}}(x) = P\{\mathbf{X} \leq x\}$

PDF

$f_{\mathbf{X}}(x) = \frac{dF_{\mathbf{X}}(x)}{dx}$

Expected value: Motivation:

Averages, Mean
Weighted average.

Expected value: Definition

Discrete random variable

$$E(X) = \sum_{\forall x} x f_X(x)$$

Continuous random variable

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

Alternate notations
$E(X) = \mu$
$f_X(x) = p(x)$

Variance

Variance of a random variable is $\sigma^2 = E[(X-\mu)^2]$

$$E[(X-\mu)^2] = E[X^2 - 2X\mu + \mu^2]$$
$$E(X) = \mu$$
$$E[(X-\mu)^2] = E(X^2) - 2\mu^2 + \mu^2$$
$$E[(X-\mu)^2] = E(X^2) - \mu^2$$

σ is called standard deviation.

Moments

Moment about origin

The k^{th} moment about origin of a random variable is defined as $E(X^k)$ where $k \in \{1, 2, \cdots\}$

$$m_k = E(X^k) = \int_{-\infty}^{\infty} x^n f_X(x) dx$$

Central moment

The k^{th} central moment of a random variable is defined as $E((X - E(X))^k)$ where $k \in \{1, 2, \cdots\}$

$$\mu_k = E((X - E(X))^k) = \int_{-\infty}^{\infty} (x - E(X))^n f_X(x) dx$$

What are the values of μ_0 and μ_1 ?

Moments

The second central moment is the variance and it indicates the spread of the density function about mean.

The third central moment is called skew and it is a measure of asymmetry of the density function about mean.

Two important inequalities

Chebychev Inequality

For a random variable X with mean μ and variance σ^2
$P(|X - \mu| \geq \epsilon) \leq \sigma^2/\epsilon^2$ for $\epsilon > 0$

Markov inequality

$P(X \geq a) \leq \mu/a$ for $a > 0$

Moment generating functions

The moment generating function of a random variable X is defined as $M_X(\nu) = E(e^{\nu X})$ where ν is a real number such that $-\infty < \nu < \infty$

Moment generating functions

What is e^x

$e^x = \sum_{n=0}^{\infty} x^n/n!$
$e^x = 1/1 + x/1 + x^2/2 + x^3/6 + \ldots$

$e^{\nu X} = 1/1 + \nu X/1 + (\nu X)^2/2 + (\nu X)^3/6 + \ldots$
MGF$= E(1/1 + \nu X/1 + (\nu X)^2/2 + (\nu X)^3/6 + \ldots$

The moments can be found out by successively differentiating the MGF with respect to ν and setting ν to 0.

Moment generating functions

If X is the out come of a fair die, find the MGF of X and using MGF find out expectation of X and variance of X

$P(X = x) = 1/6, x = \{1, \cdots, 6\}$
MGF= $\sum_{X=1}^{6} e^{\nu X} p(X = x)$
$=1/6e^{\nu} + 1/6e^{2\nu} + 1/6e^{2\nu} + 1/6e^{3\nu} + 1/6e^{4\nu} + 1/6e^{5\nu} + 1/6e^{6\nu}$

Characteristic functions

The characteristic function of a random variable X is defined as
$\Phi_X(\omega) = E(e^{j\omega X})$ where ω is a real number such that $-\infty < \omega < \infty$
$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x) e^{j\omega X} dx$

The density function can be found out by
$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\omega) d\omega$

Characteristic functions

The moments can be found out from the characteristic function by differentiating it and setting $\omega = 0$

$M_n = (-j)^n \frac{d^n \Phi_X(\omega)}{d\omega^n}|_{\omega=0}$